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Comment on path integral derivation of Schrödinger equation in spaces with curvature and torsion

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Abstract. We present a derivation of the Schrödinger equation for a path integral of a point particle in a space with curvature and torsion which is considerably shorter and more elegant than that commonly found in the literature.

1. Introduction

When studying the historic paper on path integrals in spaces with curvature and torsion by DeWitt [1], or any of its successors [2–4] one may rightfully be frustrated by the tedious algebra involved in deriving the simple Schrödinger equation satisfied by a point particle in curved space. If torsion were to be admitted to the geometry, this derivation would become even more involved.

In this paper, we want to point out that there exists a much shorter and more elegant derivation based on the use of non-holonomic coordinate transformations (mathematicians would speak of non-holonomic bases for the tangent space). By analogy with the physics of defects, these have recently been found to be an essential tool in *predicting* the correct path integral in such spaces [5–7]. In particular, they do not produce undesirable terms proportional to the scalar curvature R in the Schrödinger equation, in contrast to the findings in previous works [1–4]. The use of such transformations is very fundamental, and the transformation rules constitute a new *quantum equivalence principle* [5, 6] which extends Einstein's famous classical principle to spaces with curvature and torsion, specifying the dynamical laws of both classical [8] and quantum physics [7].

Here we do not intend to convince the reader of the virtues of this principle but want to direct his/her attention upon a useful technical advantage of using such transformations: a drastic simplification of the above-mentioned derivation of the Schrödinger equation.

2. Schrödinger equations from path integral

Let us first recall the simple derivation of the Schrödinger equations from path integral of a free non-relativistic particle of mass M in Euclidean space. The set of fluctuating paths in D dimensions is parametrized by the time-dependent Cartesian coordinates $x^i(t)$ $(i = 1, \ldots, D)$. The time axis is sliced into $N+1$ intervals (t_n, t_{n-1}) $(n = 1, \ldots, N+1)$ of

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width Δt and the positions $x^i(t_n)$ are denoted by x_n^i. If Δx_n^i are the differences $x_n^i - x_{n-1}^i$, the time-sliced path integral has the measure

$$\int \mathrm{d}\mu_x^{\mathrm{new}} := \lim_{N\to\infty}\left[\prod_{n=2}^{N+1}\int \frac{\mathrm{d}^D(\Delta x_n)}{(2\pi \mathrm{i}\overline{h}\Delta t/M)^{D/2}}\right]\exp\left[\frac{\mathrm{i}}{\overline{h}}\sum_{n=1}^{N+1} M\frac{(\Delta x_n^i)^2}{2\Delta t}\right]. \tag{1}$$

Separating the last integral over Δx_{N+1} from the others, the amplitude $\psi_t(x)$ at the time t is seen to satisfy the recursion relation

$$\psi_t(x) = \int \frac{\mathrm{d}^D(\Delta x)}{(2\pi \mathrm{i}\overline{h}\Delta t/M)^{D/2}}\exp\left[-\frac{M}{2\mathrm{i}\overline{h}\Delta t}(\Delta x^i)^2\right]\psi_{t-\Delta t}(x-\Delta x). \tag{2}$$

Expanding $\psi_{t-\Delta t}(x-\Delta x)$ in powers of Δx and using the Gaussian integral formula

$$\int \frac{\mathrm{d}^D\Delta}{(2\pi\varepsilon/a)^D}\exp\left[-\frac{a}{2\varepsilon}\left(\Delta^i\Delta^i\right)\right](1+b_i\Delta^i+b_{ij}\Delta^i\Delta^j+\ldots) = 1+\frac{\varepsilon}{a}b_{ii}+\mathcal{O}(\varepsilon^2) \tag{3}$$

one finds

$$\begin{aligned}\psi_t(x) = \int \frac{\mathrm{d}^D(\Delta x)}{(2\pi \mathrm{i}\overline{h}\Delta t/M)^{D/2}}\exp\left[\mathrm{i}\frac{M}{2\overline{h}\Delta t}(\Delta x^i)^2\right]\Bigg[\psi_{t-\Delta t}(x) - (\Delta x)^i\nabla_i\psi_{t-\Delta t}(x)\\ +\frac{1}{2}(\Delta x)^i(\Delta x)^j\nabla_i\nabla_j\psi_t(x)+\mathcal{O}((\Delta t)^{3/2})\Bigg]\\ = \psi_t(x)+\Delta t\left[-\partial_t\psi_t(x)+\frac{\mathrm{i}\overline{h}}{2M}\boldsymbol{\nabla}^2\psi_t(x)\right]+\mathcal{O}((\Delta t)^2).\end{aligned} \tag{4}$$

In the limit $\Delta t \to 0$, this yields the Schrödinger equation

$$\mathrm{i}\overline{h}\partial_t\psi = -\frac{\overline{h}^2}{2M}\Delta\psi \tag{5}$$

where $\Delta \equiv \boldsymbol{\nabla}^2$ is the Laplace operator.

In non-Euclidean spaces, the derivation becomes complicated. For the sake of generality, we shall admit some kind of curvature and torsion. Let $q^\mu(t)$ $(\mu = 1,\ldots D)$ be the paths in such a general space $\mathcal{S}_q$. A non-holonomic transformation

$$\dot{x}^i = e_\mu^i(q)\dot{q}^\mu \tag{6}$$

maps them into a reference space $\mathcal{S}_x$ of x^i-vectors $(i = 1,\ldots, D)$ with a Euclidean metric. Under this transformation, the measure of path integration (1) goes over into [7]

$$\int \mathrm{d}\mu_q = \lim_{N\to\infty}\left[\prod_{n=2}^{N+1}\int \frac{\mathrm{d}^D(\Delta q_n)}{(2\pi \mathrm{i}\overline{h}\Delta t/m)^{D/2}}\frac{\partial(\Delta x_n)}{\partial(\Delta q_n)}\right]\exp\left(\frac{\mathrm{i}}{\overline{h}}\Delta\mathcal{A}_q^{N+1}\right) \tag{7}$$

where $\Delta\mathcal{A}_q^{N+1}$ is the time-sliced version of the classical action

$$\mathcal{A}_q = \int_{t_1}^{t_2}\mathrm{d}t\frac{m}{2}g_{\mu\nu}(q)\dot{q}^\mu\dot{q}^\nu \tag{8}$$

of the system, evaluated along the classical orbits. The tensor $g_{\mu\nu}(q) = \sum_i e_\mu^i(q)e_\nu^i(q)$ is the metric with a non-zero Riemann curvature tensor $\bar{R}_{\mu\nu\lambda}{}^\kappa$ derived from the covariant curl of the usual Christoffel symbols (Levi-Civita connection). The action $\Delta\mathcal{A}_q^{N+1}$ can be expanded around prepoint, midpoint or postpoint in each time slice, with the latter being the most convenient one for the derivation of the Schrödinger equation [5, 7]. It will be denoted by $\Delta\mathcal{A}_>^{N+1}$.

By a straightforward but tedious generalization of the above Euclidean derivation one finds in the space $\mathcal{S}_q$ the Schrödinger equation [1–5, 7]

$$\mathrm{i}\hbar\partial_t\psi = -\frac{\hbar^2}{2m}\Delta\psi \tag{9}$$

where $\Delta \equiv D^\mu D_\mu$ is the Laplace operator in a general metric-affine space. This operator is related to the Laplace–Beltrami operator $\bar{\Delta} \equiv \bar{D}^\mu \bar{D}_\mu \equiv \sqrt{g}^{-1}\partial_\mu\sqrt{g}g^{\mu\nu}\partial_\nu$ by $D^\mu D_\mu = \bar{\Delta} - 2S^\mu\partial_\mu$. The symbol D_μ denotes the covariant derivative with respect to the affine connection $\Gamma_{\mu\nu}{}^\gamma = e_i{}^\gamma\partial_\mu e^i_\nu$ with non-zero torsion $S_{\mu\nu}{}^\gamma = \Gamma_{[\mu\nu]}{}^\gamma$, and zero Cartan curvature $R_{\mu\nu\lambda}{}^\kappa = 0$, whereas $\bar{D}_\alpha$ is the covariant derivative with respect to the Christoffel symbols $\bar{\Gamma}_{\mu\nu}{}^\kappa = g^{\kappa\lambda}\left(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu}\right)/2$ with non-zero Riemann curvature $\bar{R}_{\mu\nu\lambda}{}^\kappa$, and zero torsion $\bar{S}_{\mu\nu}{}^\gamma = \bar{\Gamma}_{[\mu\nu]}{}^\gamma = 0$.

What makes the historic derivation [1–4] of the Schrödinger equation (9) tedious is the fact, observed first in [9], that one has to calculate the time-sliced action $\Delta\mathcal{A}^{N+1}_>$ up to fourth-order terms in the differences $\Delta q^\mu_n = q^\mu_n - q^\mu_{n-1}$, and the Jacobian $\partial(\Delta x_n)/\partial(\Delta q_n)$ up to the second order. Since Δq^μ_n are of the order $\sqrt{\Delta t}$, all these terms contribute to first order in Δt, and thus to the Schrödinger equation. Instead of (3), the derivation requires the more general formula

$$\begin{aligned}\int \frac{\mathrm{d}^D\Delta}{(2\pi\varepsilon/a)^D}\sqrt{g}&(1 + b_\mu\Delta^\mu + b_{\mu\nu}\Delta^\mu\Delta^\nu + \cdots)\\ &\times\exp\left[-\frac{a}{2\varepsilon}(g_{\mu\nu}\Delta^\mu\Delta^\nu + G_{\mu\nu\lambda}\Delta^\mu\Delta^\nu\Delta^\lambda + G_{\mu\nu\lambda\kappa}\Delta^\mu\Delta^\nu\Delta^\lambda\Delta^\kappa + \cdots)\right]\\ &= 1 + \frac{\varepsilon}{a}\left[b_{\mu\nu}g^{\mu\nu} - \frac{1}{2}(G_{\mu\nu\lambda\kappa} + G_{\mu\nu\lambda}b_\kappa)g^{\mu\nu\lambda\kappa} + \frac{1}{8}G_{\alpha\beta\gamma}G_{\mu\nu\lambda}g^{\alpha\beta\gamma\mu\nu\lambda}\right] + \mathcal{O}(\varepsilon^2).\end{aligned} \tag{10}$$

Here $b_\mu, b_{\mu\nu}$, and $g_{\mu\nu}, G_{\mu\nu\lambda}, G_{\mu\nu\lambda\kappa}, \ldots$ are ε-independent coefficients, g denotes the determinant of the metric $\det\|g_{\mu\nu}\|$, and $g^{\mu_1,\ldots,\mu_{2n}}$ are symmetric tensors formed from products of n metric tensors: $g^{\mu\nu\lambda\kappa} = g^{\mu\nu}g^{\lambda\kappa} + g^{\mu\lambda}g^{\nu\kappa} + g^{\mu\kappa}g^{\nu\lambda}$, $g^{\alpha\beta\gamma\mu\nu\lambda} = g^{\alpha\beta}g^{\gamma\mu\nu\lambda} + g^{\alpha\gamma}g^{\beta\mu\nu\lambda} + \cdots$. In the course of the calculations one encounters many cancellations, which make the final result (9) again very simple. There must be a derivation which reflects the simplicity of the final result from the beginning, and this is what we want to present now.

3. New derivation

First we observe that the time-sliced action $\Delta\mathcal{A}^{N+1}_q$ which consists of the $\sum_{n=2}^{N+1}\Delta\mathcal{A}_n$ calculated along the classical trajectories is a simple Gaussian when expressed in terms of the velocities $\dot{q}^\mu$ in each time slice. The classical trajectories are described by the equation of motion

$$\ddot{q}^\mu + \Gamma_{\nu\lambda}{}^\mu\dot{q}^\nu\dot{q}^\lambda = 0. \tag{11}$$

This equation implies the conservation of the energy $E = g_{\mu\nu}\dot{q}^\mu\dot{q}^\nu/2$. As a consequence, the short-time action is

$$\Delta\mathcal{A} = \frac{m}{2}g_{\mu\nu}\dot{q}^\mu\dot{q}^\nu\Delta t = \frac{m}{2\Delta t}g_{\mu\nu}\Delta\xi^\mu\Delta\xi^\nu \tag{12}$$

where we have found it useful to introduce the *vector* quantities $\Delta\xi^\mu$ defined by $\Delta\xi^\mu \equiv \dot{q}^\mu\Delta t$ whose size is of the order of $(\Delta t)^{1/2}$, to have quantities comparable to the previous differences Δx^i. In (12), the velocities $\dot{q}^\mu$ as well as the metric $g_{\mu\nu}$ are calculated at the

latest time t_n in the interval (t_n, t_{n-1}), to have the preferred post-point form (any time would give the same result, due to energy conservation).

An explicit *functional* relation between $\Delta\xi^\mu$ and Δq^μ is obtained by Taylor-expanding

$$\begin{aligned}\Delta q^\mu &= q^\mu(t_n) - q^\mu(t_{n-1}) = q^\mu(t_n) - q^\mu(t_n - \Delta t) \\ &= \dot q^\mu \Delta t - \frac{(\Delta t)^2}{2!}\ddot q^\mu + \frac{(\Delta t)^3}{3!}\dddot q^\mu + \mathcal{O}((\Delta t)^4)\end{aligned} \tag{13}$$

and using (11), which implies for the higher time derivatives

$$\ddot q^\mu = -\Gamma_{\nu\lambda}{}^\mu \dot q^\nu \dot q^\lambda \qquad \dddot q^\mu = -(\partial_\kappa \Gamma_{\nu\lambda}{}^\mu - 2\Gamma_{\{\kappa\delta\}}{}^\mu \Gamma_{\nu\lambda}{}^\delta)\dot q^\kappa \dot q^\nu \dot q^\lambda$$

where curly braces around indices indicate their symmetrization. Hence,

$$\Delta q^\mu = \Delta\xi^\mu + \frac{1}{2!}\Gamma_{\nu\lambda}{}^\mu \Delta\xi^\nu \Delta\xi^\lambda - \frac{1}{3!}(\partial_\kappa \Gamma_{\nu\lambda}{}^\mu - 2\Gamma_{\{\kappa\delta\}}{}^\mu \Gamma_{\nu\lambda}{}^\delta)\Delta\xi^\kappa \Delta\xi^\nu \Delta\xi^\lambda + \cdots \tag{14}$$

and this may be inverted to

$$\Delta\xi^\mu = \Delta q^\mu - \frac{1}{2!}\Gamma_{\nu\lambda}{}^\mu \Delta q^\nu \Delta q^\lambda + \frac{1}{3!}(\partial_\kappa \Gamma_{\nu\lambda}{}^\mu + \Gamma_{\{\kappa\delta\}}{}^\mu \Gamma_{\nu\lambda}{}^\delta)\Delta q^\kappa \Delta q^\nu \Delta q^\lambda + \cdots. \tag{15}$$

Using this equation, we change the integration variables Δq_n^μ in formula (7) into $\Delta\xi_n^\mu$, and find the following measure for the path integral

$$\int \mathrm{d}\mu_q = \lim_{N\to\infty}\left[\prod_n \frac{\mathrm{d}^D \Delta\xi_n}{(2\pi \mathrm{i}\hbar\Delta t_n/m)^{D/2}}\sqrt{g(q_n)}\right]\exp\left[\frac{\mathrm{i}}{\hbar\Delta t}\sum_{n=1}^{N+1} g_{\mu\nu}(q_n)\Delta\xi_n^\mu \Delta\xi_n^\nu\right]. \tag{16}$$

This expression is related to the flat-space measure (1) by just a linear transformation. The reason for this is that in terms of the nonholonomic x-variables, the equation of motion (11) is trivial: $\ddot x^i = 0$. Hence $\dot x^i =$ constant, and

$$\Delta x^i = \dot x^i \Delta t = e_\mu^i \dot q^\mu \Delta t = e_\mu^i \Delta\xi^\mu \tag{17}$$

so that $\partial(\Delta x)/\partial(\Delta\xi) = \det\|e_\mu^i\| = \sqrt{g}$.

Using the simple measure (16), we immediately find for an amplitude $\psi_t(q)$ the recursion relation (again by removing the last slice from the product of integrals)

$$\psi_t(q) = \int \frac{\mathrm{d}^D \Delta\xi}{(2\pi \mathrm{i}\hbar\Delta t/M)^{D/2}}\sqrt{g}\exp\left(-\frac{M}{2\,\mathrm{i}\hbar\Delta t}g_{\mu\nu}\Delta\xi^\mu \Delta\xi^\nu\right)\psi_{t-\Delta t}(q - \Delta q(\Delta\xi)). \tag{18}$$

Expanding the amplitude inside the integral in powers of Δq^μ, and expressing these in terms of $\Delta\xi^\mu$ with the help of (14), we find

$$\begin{aligned}\psi_t(q) = \int \frac{\mathrm{d}^D \Delta\xi}{(2\pi \mathrm{i}\hbar\Delta t/M)^{D/2}}\sqrt{g}\exp\left(-\frac{M}{2\,\mathrm{i}\hbar\Delta t}g_{\mu\nu}\Delta\xi^\mu \Delta\xi^\nu\right)\Bigg[\psi_{t-\Delta t}(q) \\ -\left(\Delta\xi^\mu + \frac{1}{2}\Gamma_{\nu\lambda}{}^\mu \Delta\xi^\nu \Delta\xi^\lambda\right)\partial_\mu \psi_{t-\Delta t}(q) \\ +\frac{1}{2}\Delta\xi^\mu \Delta\xi^\nu \partial_\mu \partial_\nu \psi_t(q) + \mathcal{O}((\Delta t)^{3/2})\Bigg] \\ = \psi_t(q) + \Delta t\left\{-\partial_t \psi_t(q) + \frac{\mathrm{i}\hbar}{2m}[g^{\mu\nu}\partial_\mu \partial_\nu \psi_t(q) - \Gamma_\kappa{}^{\kappa\mu}\partial_\mu \psi_t(q)]\right\} + \mathcal{O}((\Delta t)^2).\end{aligned} \tag{19}$$

After making use of the identity $g^{\mu\nu}\partial_\mu\partial_\nu - \Gamma_\kappa{}^{\kappa\mu}\partial_\mu = D^\mu D_\mu$, this is precisely the Schrödinger equation (9).

4. Comparison with the historic derivation of the Schrödinger equation in spaces with curvature and torsion

To compare the present derivation with the historic derivation in [1–5, 7], we must calculate $\Delta\mathcal{A}$ as a function of Δq^μ up to fourth-order terms using formulae (12) and (15). This yields

$$\Delta\mathcal{A} = \frac{m}{2\Delta t}\Bigg[g_{\mu\nu}\Delta q^\mu\Delta q^\nu - \bar\Gamma_{\mu\nu\lambda}\Delta q^\mu\Delta q^\nu\Delta q^\lambda + \frac{1}{3}\left(\partial_\mu\bar\Gamma_{\nu\lambda\kappa} - \frac{1}{4}\bar\Gamma_{\mu\nu\sigma}\bar\Gamma_{\lambda\kappa}{}^\sigma\right)\Delta q^\mu\Delta q^\nu\Delta q^\lambda\Delta q^\kappa + \frac{1}{3}S^\sigma{}_{\mu\nu}S_{\sigma\lambda\kappa}\Delta q^\mu\Delta q^\nu\Delta q^\lambda\Delta q^\kappa + \mathcal{O}((\Delta q)^5)\Bigg]. \tag{20}$$

The Jacobian $\partial(\Delta x)/\partial(\Delta q)$ is conveniently given in the exponential form

$$\partial(\Delta x)/\partial(\Delta q) = \sqrt{g}\exp(\mathrm{i}\Delta\mathcal{A}'_J/\hbar) \tag{21}$$

with an effective action

$$\frac{\mathrm{i}}{\hbar}\Delta\mathcal{A}'_J = -\Gamma_{\{\mu\nu\}}{}^\mu\Delta q^\nu + \frac{1}{2}\Bigg[\partial_{\{\mu}\Gamma_{\lambda\nu\}}{}^\mu + \frac{1}{3}\left(\Gamma_{\{\mu\kappa\}}{}^\mu\Gamma_{\{\lambda\nu\}}{}^\kappa - \Gamma_{\{\lambda\kappa\}}{}^\mu\Gamma_{\{\nu\mu\}}{}^\kappa\right)\Bigg]\Delta q^\lambda\Delta q^\nu + \mathcal{O}((\Delta q)^3). \tag{22}$$

Inserting these expansions into the product of integrals (7) we obtain a large number of terms. Upon applying formula (10), most of these cancel each other, leading again to the Schrödinger equation (9).

The additional labour is the same as if we were to prove the obvious identity

$$[\partial(\Delta x)/\partial(\Delta q)][\partial(\Delta q)/\partial(\Delta\xi)] = \partial(\Delta x)/\partial(\Delta\xi) = \sqrt{g} \tag{23}$$

by writing $[\partial(\Delta x)/\partial(\Delta q)]$ as in (21) expanded via (22), and writing $\partial(\Delta q)/\partial(\Delta\xi) = \exp(\mathrm{i}\Delta\mathcal{A}''_J/\hbar)$ with

$$\frac{\mathrm{i}}{\hbar}\Delta\mathcal{A}''_J = \Gamma_{\{\mu\nu\}}{}^\mu\Delta\xi^\nu - \frac{1}{2}(\partial_{\{\mu}\Gamma_{\lambda\nu\}}{}^\mu - 2\Gamma_{\{\{\mu\kappa\}}{}^\mu\Gamma_{\lambda\nu\}}{}^\kappa + \Gamma_{\{\mu\lambda\}}{}^\kappa\Gamma_{\{\kappa\nu\}}{}^\mu)\Delta\xi^\lambda\Delta\xi^\nu + \mathcal{O}(\Delta\xi^3) \tag{24}$$

which follows from (14). Inserting here (15) one may re-express the right-hand side as a power series in Δq, and form the sum $\Delta\mathcal{A}'_J + \Delta\mathcal{A}''_J$. The many terms in this sum all cancel each other, as required by (23).

All such complications are avoided by working with the variables $\Delta\xi^\mu$ in which the exponential in (19) is a pure Gaussian.

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Note added in proof. After this paper was accepted, Professor J S Dowker drew our attention to an early work of his in 1974 *J. Phys. A: Math. Gen.* **7** 1256, in which he simplifies the derivation of the Schrödinger equation in a way similar to ours. His mapping, however, is different and produces undesirable *R*-terms as in previous works [1–4].

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